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Bounds on Correlation Decay for Long-Range Vector Spin Glasses

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We give upper bounds on the decay of correlation functions for long-range $SO(N)$-symmetric spin-glass models in one and two dimensions using McBryan–Spencer techniques. In doing so we extend recent results of Picco.

KEY WORDS: Vector spin glass; long range; McBryan–Spencer bounds; stochastic correlation decay.

1. INTRODUCTION

Spin-glass models are generally constructed by taking random interactions with zero average. The occurrence of both ferro- and antiferromagnetic terms can give rise to cancellation effects. In particular when the interactions are independent, it is known that their effective range tends to be shorter than in the corresponding nonrandom case. Examples of this phenomenon are the existence of the free energy density for interactions which are square summable but not absolutely summable,[1–3] and the absence of symmetry breaking in one- and two-dimensional models[4–10] for which the corresponding nonrandom, ferromagnetic, models have a first-order transition.[11–13] In this paper we address the case of N-vector spins in one and two dimensions, where it is already known that there is no symmetry breaking.

We will indicate how fast the pair correlation functions decrease to zero. This problem has already been treated by Picco,[7] but using some ideas from former work,[8,9] we are able to extend his results considerably. For the detailed estimates we use, similarly to Picco, ideas of Messager,

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Miracle-Sole, and Ruiz[14] which have been developed to prove McBryan–Spencer bounds for nonrandom models with interactions of which the range becomes as long as possible.

2. THE ONE-DIMENSIONAL CASE

Let us consider Hamiltonians of the form

$$H = - \sum_{i,j \in \mathbb{Z}^d} J(i,j)\, |i-j|^{-\alpha}\, \mathbf{s}_i \mathbf{s}_j \tag{1}$$

where the $J(i, j)$ are independent, identically distributed random variables with a distribution which has bounded support and for which $\mathbb{E}J(i, j) = 0$ and the s_i are plane rotors (two-component vectors). It is convenient to write

$$\tilde{J}(i,j) = J(i,j)\, |i-j|^{-\alpha} \tag{2}$$

In this section we treat the one-dimensional case for which the proof works somewhat more smoothly and for which we get a power law decay with probability one (i.e., for almost all J configurations).

Theorem 1. If $d = 1$ and $\alpha > 1$, there is a positive random variable $f(\{J\})$ which is finite J-almost surely, such that

$$|\langle \mathbf{s}_0 \mathbf{s}_N \rangle(\{J\})| \leqslant f(\{J\})\, N^{-(\alpha - 1/2)}$$

Proof. According to McBryan and Spencer,[15,16] for all $\{a_i\}$

$$|\langle \mathbf{s}_0 \mathbf{s}_N \rangle| \leqslant \frac{Z(H')}{Z(H)} \exp[-(a_N - a_0)] \tag{3}$$

where $Z(H)$ is the partition function corresponding to H,

$$H' = -\sum_{i,j} \tilde{J}'(i,j)\, \mathbf{s}_i \mathbf{s}_j \tag{4a}$$

and

$$\tilde{J}'(i,j) = \cosh(a_i - a_j)\, \tilde{J}(i,j) \tag{4b}$$

The variational principle[17] implies

$$\frac{Z(H')}{Z(H)} \leqslant \exp(-\langle H' - H \rangle_{H'}) \tag{5}$$

where $\langle \cdot \rangle_{H'}$ denotes the thermal expectation with respect to H'. This expression will be finite J-almost surely if

$$\mathbb{E}\, |\langle H' - H \rangle_{H'}| < \infty \tag{6}$$

where (6) has to be understood uniformly in the volume. Technically, the proof of (6) is very similar to the estimate on the relative entropy in Ref. 9. Following Messager, Miracle-Sole, and Ruiz (to be referred as MMR),[14] we choose $a_j = a_{|j|}$ such that for $j > 0$ $a_j - a_{j-1} = K/j$. Then

$$(a_N - a_0) \sim K \ln N \tag{7}$$

To prove (6), we firstly note that just as in Ref. 9, we only have to consider the tail interactions (the short-range part can be treated along the lines of MMR). Hence we can assume that all the $J'(i, j)$ are small, whenever $K < \alpha$.

By the Cauchy–Schwartz inequality

$$\mathbb{E}\, |\langle H - H' \rangle_{H'}| \leqslant (\mathbb{E}\langle H - H' \rangle_{H'}^2)^{1/2} \tag{8}$$

Now $\mathbb{E}\langle H - H' \rangle_{H'}^2$ is a sum of terms of the form

$$\mathbb{E}\langle \mathbf{s}_i \mathbf{s}_j \rangle_{H'} \langle \mathbf{s}_k \mathbf{s}_l \rangle_{H'} [\tilde{J}'(i, j) - \tilde{J}(i, j)][\tilde{J}'(k, l) - \tilde{J}(k, l)] \tag{9}$$

We rewrite

$$\langle \mathbf{s}_i \mathbf{s}_j \rangle_{H'} = \frac{\langle \mathbf{s}_i \mathbf{s}_j \exp[\tilde{J}'(i, j)\, \mathbf{s}_i \mathbf{s}_j + \tilde{J}'(k, l)\, \mathbf{s}_k \mathbf{s}_l] \rangle_{H'_{ij,kl}}}{\langle \exp[\tilde{J}'(i, j)\, \mathbf{s}_i \mathbf{s}_j + \tilde{J}'(k, l)\, \mathbf{s}_k \mathbf{s}_l] \rangle_{H'_{ij,kl}}} \tag{10}$$

and do the same with $\langle \mathbf{s}_k \mathbf{s}_l \rangle_{H'}$

The modified thermal average $\langle \cdot \rangle_{H'_{ij,kl}}$ is obtained by removing the two interactions between the spins at i and j, c.q. at k, and l from the Hamiltonian. The advantage of using modified thermal averages is, that one may perform an average over the $J(i, j)$ and $J(k, l)$ when they are on the outside of the modified thermal average. It turns out that this allows one to extend the case $\alpha > 3/2$[4-7] to $\alpha > 1$[8-10]. If $i, j = k, l$

$$(9) \leqslant |i - j|^{-2\alpha} [\cosh(a_i - a_j) - 1]^2 \tag{11}$$

If $i, j \neq k, l$ we can develop (9) into an absolutely convergent Taylor expansion[9] in $\tilde{J}'(i, j)\, \mathbf{s}_i \mathbf{s}_j + \tilde{J}'(k, l)\, \mathbf{s}_k \mathbf{s}_l$, making use of the representation (10), and we can term for term perform the average over $J(i, j)$ and $J(k, l)$.

Owing to the fact that $\mathbb{E}J(i, j)=0$, all the first-order terms disappear and we get

$$(9) \leqslant C_1 \, |i-j|^{-2\alpha} \, |k-l|^{-2\alpha} [\cosh^2(a_i - a_j) - \cosh(a_i - a_j)] \\ \times [\cosh^2(a_k - a_l) - \cosh(a_k - a_l)] \tag{12}$$

We notice that for large x $\cosh^2 x \sim \cosh 2x$.

Now we are able to apply the one-dimensional estimates of MMR. For the proofs and details we refer to Ref. 14. For the diagonal terms in $\mathbb{E}\{\sum \langle [\tilde{J}'(i,j) - \tilde{J}(i,j)] \, \mathbf{s}_i \mathbf{s}_j \rangle_{H'}\}^2$ we have, using (11)

$$\sum_{i,j} |i-j|^{-2\alpha} [\cosh(a_i - a_j) - 1]^2 \leqslant C_2 \sum_{n=0}^{\infty} Q_\alpha(n) \tag{13}$$

where

$$\begin{aligned} Q_\alpha(n) &= \sum_{m=1}^{\infty} m^{-2\alpha} [\cosh(a_{n+m} - a_n) - 1]^2 \\ &= \sum_{m=1}^{n} \cdots + \sum_{n+1}^{\infty} \cdots = S_\alpha(n) + T_\alpha(n) \end{aligned} \tag{14}$$

The short-range part $S_\alpha(n)$ is estimated by using $\cosh x - 1 = O(x^2)$ for small x and the tail part $T_\alpha(n)$ by $(\cosh x - 1)^2 = O(e^{2x})$ for large x.

The estimates of MMR imply that (13) is finite whenever $2K \leqslant 2\alpha - 1$ or $K \leqslant \alpha - 1/2$. For the nondiagonal terms, using (12) we get an upper bound of the form

$$\left[C_2 \sum_{n=0}^{\infty} Q_\alpha(n) \right]^2 \tag{15}$$

which is also finite, because of the same estimate. This finishes the proof of (6) and hence of the theorem. ▮

Remarks. (1) The extension to general N-vector models is straightforward. One can for example use the theorem for the first two directions and then use the fact that there is no breaking of the $SO(N)$ symmetry to extend the result to the other $N-1$ directions.

(2) Unbounded distributions of the $J'(i, j)$ can be treated too, as in Ref. 9.

(3) For the nonrandom case the McBryan–Spencer bounds give $\langle \mathbf{s}_0 \mathbf{s}_N \rangle \lesssim N^{-(\alpha-1)}$ which is not an optimal value, since one has $\langle \mathbf{s}_0 \mathbf{s}_N \rangle \sim N^{-\alpha}$ in this case (the correlations decay at the same rate as the

interaction).[18,19] For random models this kind of results have been obtained in the case $\alpha > 3/2$,[5] but for the case $3/2 > \alpha > 1$ it is not known at present whether it is possible to improve on the McBryan–Spencer bounds.

3. THE TWO-DIMENSIONAL CASE

In two dimensions the reduction of an estimate for a random interaction with decay power α to an estimate for a nonrandom interaction with decay power 2α is also possible. However the precise results are weaker in this case.

Theorem 2. Let $d = 2$, $\alpha > 2$, and $\gamma < 1$. Then for all $K > 0$ there exists a sequence of random variables F_N, such that

$$-\ln |\langle \mathbf{s}_0 \mathbf{s}_N \rangle| \geqslant F_N$$

and

$$\lim_{N \to \infty} \frac{F_N}{(\ln N)^\gamma} = K \qquad \text{in } L^2\text{-sense}$$

Proof. We take for the $\{a_j\}$ in the McBryan–Spencer estimate (3) the choice of Picco, $a_{|j|} - a_{|j|-1} = \bar{K}/j(\ln^+ j)^{1-\gamma}$, $|j| = 1,..., N$.

Here $|j| = j$ for all points on the boundary of the square with sides $2j \times 2j$ which has the origin at its center. With this choice

$$(a_N - a_0) \sim C_3 \bar{K} (\ln N)^\gamma \tag{16}$$

Instead of (6) we derive in this case (see below)

$$\mathbb{E}\, |\langle H - H'_N \rangle_{H'_N}| \leqslant (\mathbb{E} \langle H - H'_N \rangle^2_{H'_N})^{1/2} \leqslant O((\ln N)^{2\gamma - 1}) \tag{17}$$

From (16) and (17) we obtain by choosing $\bar{K}$ suitably

$$\lim_{N \to \infty} \frac{(a_N - a_0) - (\mathbb{E} \langle H - H'_N \rangle^2_{H'_N})^{1/2}}{(\ln N)^\gamma} = K$$

which gives the theorem.

To derive (17) we apply exactly the same procedure as in the proof of Theorem 1 which replaces α by 2α in the estimates. The appropriate MMR estimate is a variation on the estimates on p. 92 of Ref. 14. Equation (13) becomes replaced by an upper bound of the form $\sum_{n=1}^{N} n Q_{\mu,\gamma}(n)$ (and equation (15) by an upper bound of the form $[\sum n Q_{\mu,\gamma}(n)]^2$).

For the short-range part one obtains, using the notation of MMR,

$$S_{\mu,\gamma}(n)=\sum_{m=1}^{n(\ln^+ n)^{1-\gamma}} \frac{1}{m^\mu}\left\{\cosh\left[\frac{K}{n(\ln^+ n)^{1-\gamma}}+\cdots\right.\right.$$
$$\left.\left.+\frac{K}{(n+m-1)[\ln^+(n+m-1)]^{1-\gamma}}\right]-1\right\}$$
$$\leqslant \sum_{m=1}^{n(\ln^+ n)^{1-\gamma}} \frac{b_K K^2}{[n(\ln^+ n)^{1-\gamma}]^2}\, m^{2-\mu}=O\left(\frac{1}{n^2(\ln^+ n)^{2-2\gamma}}\right) \tag{19}$$

$$\sum_{n=1}^{N} nS_{\mu,\gamma}(n)\leqslant O(\ln N)^{2\gamma-1} \tag{20}$$

for $\mu>3$ which corresponds to $2\alpha>4$ or $\alpha>2$. The tail terms T, similar to MMR Section 3 satisfy

$$\sum_{n=1}^{\infty} nT_{\mu,\gamma}(n)<\infty \tag{21}$$

Note here that $|i-j|^{-\alpha}\cosh(a_i-a_j)\leqslant|i-j|^{-(\alpha-\varepsilon)}$ asymptotically when $|i-j|\to\infty$ for all $K>0$ and $\varepsilon>0$.

Equations (20) and (21) play the same role in this case as the finiteness of $\sum_{n=1}^{\infty} Q_\alpha(n)$ [compare (13) and (14)] in the one-dimensional case. Together they give (17) and hence the theorem. ∎

Remark. In fact the bounds (19) and (20) can be improved by taking $(\cosh x-1)^2\approx O(x^4)$, but this is not possible for the nondiagonal terms, as one can see from (12).

Comments. In the two-dimensional case the result is somewhat weaker than one might hope for. Firstly the upper bound is not an almost sure property, and secondly we obtain not a power law decay but only an "almost power law" which holds true, however, at all temperatures. If we try to get a power law decay by putting in $\gamma=1$, we only get an upper bound on the correlation decay $\ln|\langle \mathbf{s}_0\mathbf{s}_N\rangle|$ for which the average over the $J(i,j)$ converges. This does not exclude that with positive probability there is no decay at all. This upper bound gives a decay power which is proportional to $(1/\beta)^2$at low temperatures (or equivalently high β). To get a stochastic convergence for the upper bound in some sense, we have to settle for something weaker than a power law.

If we compare our results with those of Picco,[7] we extended (compare Refs. 8 and 9) his treatment from the case $\alpha>3/2d$ to $\alpha>d$. Moreover we obtained an explicit, T-independent value for the upper bounds on the correlation decay, which generalizes easily to N-vector models. However, in the two-dimensional plane rotor case Picco obtains a probability one result, whereas our Theorem 2 shows convergence only in an L^2 sense.

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